D-Optimal Saturated $4 \times 4 \times m_3$ Factorial Designs

CHATZOPOULOS, A. Stavros[a]

[a] *Department of Statistics, School of Economic Sciences, University of Western Macedonia, Grevena 51100, Greece*

ABSTRACT

The objective of this paper is to give saturated $4 \times 4 \times m_3$, $m_3 \geq 4$ designs, minimizing the generalized variance of the main effects and the general mean. First, introductory concepts regarding saturated designs and the D-optimal criterion mentioned and useful notations, remarks, and lemmas presented. In the main results section, the upper bounds of the value of the determinant of D-optimal saturated $4 \times 4 \times 4$ and $4 \times 4 \times 5$ founded and the designs that achieve these bounds are presented. For the saturated $4 \times 4 \times m_3$, $m_3 \geq 6$ designs, the upper bound of the determinant is given and we present a design for which the value of D-efficiency is $99, 5\%$.

KEYWORDS

D-optimal designs, main effect designs, saturated designs

1. Introduction

The problem of finding saturated designs, under different types of optimality criteria, preoccupies many researchers the last years. [10], [9], [4] showed that all $m_1 \times m_2$ saturated main effect designs are equivalent with respect to D-optimality. [12] concluded in the same result for the symmetric m^2 factorials; they also worked on a subclass of m^3 factorials. [11] and [1] considered optimality results on almost saturated main effect designs in the two-factor case.

The extension of these findings to the three-factor case considered first by [2]; they also derived D-optimal $2 \times m_2 \times m_3$ saturated main effect designs. In the same result concluded [3] and [4]. [3] using techniques from graph theory and combinatorics, made a conjecture about the D-optimal $3 \times m_2 \times m_3$ factorials when $m_2 = 2k+1$, while [4] gave some bounds for the $3 \times m_2 \times m_3$ factorials and the designs attaining these bounds where this is feasible, using matrix manipulations.[7] using Graph theory and Combinatorics derived the maximum determinant of the associated 0-1 matrix in D-Optimal saturated main effect plans for $3 \times s_2 \times s_3$ factorials, while [8] considered the incidence (bipartite) graph, which is associated with a $3 \times s^2$ saturated main effect factorial design and using paths determine the upper bound of the value of the determinant of the adjacency matrix for each $s \geq 3$. [5] studied $2 \times 2 \times (s - k) \times s$ Resolution III saturated factorial design in order to obtain D-optimal designs.

CONTACT Author[a]. Email:schatzopoulos@uowm.gr

As the number of factors and the number of levels of factors increases the problem of finding D-optimal saturated designs becomes more difficult. This paper is the first attempt to find D-optimal $4 \times 4 \times m_3$, where $m_3 \geq 4$ designs. We present the saturated $4 \times 4 \times 4$ and $4 \times 4 \times 5$ D-optimal designs, while for the saturated $4 \times 4 \times m_3$, $m_3 \geq 6$ the given design is conjectured as D-optimal.

2.   Problem Statement and Preliminaries

Following Chatterjee and Mukerjee [2] we consider the setup of a three factor saturated experiment, involving three factors F_1, F_2 and F_3 appearing at m_1, m_2 and m_3 levels respectively, where $m_3 \geq m_2 \geq m_1 \geq 2$, with $N = m_1 + m_2 + m_3 - 2$ runs. Let the levels of F_i, $i = 1, 2, 3$ be denoted by τ_i and coded as $0, 1, \ldots, m_i - 1$, $i = 1, 2, 3$. In a complete experiment the number of the lexicographically ordered treatment combinations $\tau_1 \tau_2 \tau_3$ is $u = m_1 m_2 m_3$.

The model and the notation presented below are in the spirit of [11]. Consider an experiment, where each treatment combination appears at more than once. The usual fixed-effects model, under the absence of interactions, is given by: $Y = W\beta + \epsilon$, where Y is the response vector of the experiment and ϵ is the vector of uncorrelated random errors with zero mean and the same variance σ^2. The vector of unknown parameters is given by $\beta = (\mu, \beta_1', \beta_2', \beta_3')'$, where μ is the unknown general mean and the elements of the $(m_i - 1)\mathrm{x}m_i$ vectors β_i are unknown parameters representing a full set of mutually orthogonal contrasts belonging to main effects F_i. Note that W is the $N\mathrm{x}(m_1 + m_2 + m_3 - 2)$ model matrix, where N denotes the number of observations and A' denotes the transpose of a matrix . In a saturated design $N = m_1 + m_2 + m_3 - 2$.

The purpose of this paper is to find D-optimal resolution III designs in the class D of saturated $4 \times 4 \times m_3$, $m_3 \geq 4$ factorial designs. This means that we have to maximize the determinant of the information matrix. According to [11], for the saturated designs this is equivalent to maximize the quantity $|\det(U_d)|$, where the $N \times N$ square matrix U_d with $N = m_3 + 6$ derives from matrix W with simple matrix manipulations and can be written as:

$$U_d = \begin{pmatrix} Z_1^{(1)} & Z_2^{(1)} & Z_3 \end{pmatrix}. \tag{2.1}$$

Any row of the $N\mathrm{x}3$ matrices $Z_1^{(1)}$ and $Z_2^{(1)}$ is either null if the i-th factor enters the experiment at level 0, or equal to the i-th row of the 3×3 identity matrix if the i-th factor enters the experiment at level j, $j = 1, 2, 3$. In addition, any row of the $N \times m_3$ matrix Z_3 is equal to a row of the $m_3 \times m_3$ identity matrix. The s-th row of I_{m_3} denotes that the third factor enters the experiment at the $(s - 1)$ level, $s = 1, 2, \ldots, m_3$.

Notation 2.1. Let n_i^s, $i = 1, 2, 3$, $s = 0, 1, \ldots m_i - 1$ denote the number of runs, where the i-th factor enters the experiment at level s. Then:

$$N \quad = \quad \sum_{s=0}^{m_i - 1} n_i^s \tag{2.2}$$

If $n_i^p = k$, $i = 1, 2$, $p = 1, 2, 3$, then the p-th column of matrix $Z_i^{(1)}$ has k $1'$s, while if $n_3^{s-1} = t$, $s = 1, 2, \ldots, m_3$ the $s - th$ column of matrix Z_3 has t $1'$s.

Remark 2.2. For the saturated $m_1 \times m_2 \times m_3$, $m_3 \geq m_2 \geq m_1$ design d it holds that $n_3^s = 1$ at least $m_3 - (m_1 + m_2 - 2)$ times if $m_3 - (m_1 + m_2 - 2) > 0$. Hence, $|\det(U_d)| = |\det(U_{d'}|$, where d' is a saturated $m_1 \times m_2 \times (m_1 + m_2 - 2)$ design [4].

Applying Remark 2.2 for $m_1 = m_2 = 4$ we have that, in order to find the D-optimal $4 \times 4 \times m_3$, $m_3 \geq 6$ designs we have to find the D-optimal $4 \times 4 \times 6$ design. The proof of the following Lemma is available in [10], [6] as Theorem 6.6.1 and [4].

Lemma 2.3. *For the D-optimal saturated $m_1 \times m_2$ designs e, $m_2 \geq m_1 \geq 2$ with $N = m_1 + m_2 - 1$ runs it holds that $|det(U_e)| = 1$.*

Hereafter evaluating $|\det(U_d)|$, quantity $|\det(U_e)|$ will be omitted from the manipulations.

Lemma 2.4. *For any saturated $m_1 \times m_2 \times m_3$ design d it holds that $|det(U_d)| \leq \prod_{s=1}^{m_i-1} n_i^s$, $i = 1, 2$.*

Proof. The proof arises immediately applying Lemma 2.3 of [4] for $k = 3$ and recalling that $|\det(U_e)| = 1$ as reported by Lemma 2.3. ∎

The proof of the following Lemma is available in [4] as Lemma 2.4.

Lemma 2.5. *The maximum of the quantity $Q = \prod_{s=1}^{m_1-1} n_1^s$, subject to $n_1^0 + n_1^1 + \cdots + n_1^{m_1-} = N$ and all n_1^s, $s = 0, 1, \ldots, m_1 - 1$ positive integers is achieved when $n_1^0 \geq n_1^1 \geq \cdots \geq n_1^{m_1-1} \geq n_1^0 - 1$.*

Conclusion 2.6. From the above Lemmas and Remark 2.2 we conclude that in order to find the D-optimal $4 \times 4 \times m_3$, $m_3 \geq 6$ saturated designs it must hold $n_1^0 \geq n_1^1 \geq \cdots \geq n_1^{m_1-1} \geq n_1^0 - 1$ and $n_i^s \geq 2$ for $i = 1, 2, 3$ and $s = 0, 1, \ldots, m_1 - 1$.

Remark 2.7. If $n_i^0 = n_i^s$ for some $i = 1, 2, 3$ and $0 \leq s \leq m_i - 1$, then the quantity $|\det(U_d)|$ is left invariant by interchanging levels 0 and s.

Hence, for the D-optimal $4 \times 4 \times m_3$, $m_3 \geq 6$ designs, if $n_1^0 < m_2$, recalling Remark 2.7, we can always create at least one column in matrix $Z_2^{(1)}$ with all elements zero at the first n_1^0 rows of U_d, by interchanging the appropriate levels (usually 0 and 1). Note that the number of 1's which this column contains at the rest $N - n_1^0$ rows is greater or equal to 2. For convenience let this column be the first. Similarly, if $n_1^0 < m_3$, one can easily verify that matrix Z_3 has at least $m_3 - n_1^0 = u$ columns with all elements zero at the first n_1^0 rows of U_d and at the rest $N - n_1^0$ rows each column contains n_3^s 1's. Note that matrix Z_3 is a $N \times m_3$ matrix with rows the rows of I_{m_3}. Without loss of generality (w.l.g.) let these columns be the first of matrix Z_3. So, we can partition matrix U_d as:

$$
U_d = \left(\begin{array}{ccc|c|cc}
(0 \quad 0 \quad 0)_{n_1^0} & \mathbf{0}_{n_1^0 \times 1} & & \mathbf{0}_{n_1^0 \times u} & \\
(1 \quad 0 \quad 0)_{n_1^1} & Z_{2,12} & & Z_{3,12} & \\
(0 \quad 1 \quad 0)_{n_1^2} & Z_{2,13} & Z_{2,2} & Z_{3,13} & Z_{3,2} \\
(0 \quad 0 \quad 1)_{n_1^3} & Z_{2,14} & & Z_{3,14} &
\end{array} \right), \qquad (2.3)
$$

where matrices $Z_{2,1j}$ are $n_1^{j-1} \times (m_2 - 1)$, $j = 2, 3, 4$, matrices $Z_{3,1j}$ are $n_1^{j-1} \times u$, $j = 2, 3, 4$ and $Z_{2,2}$ and $Z_{3,2}$ are $N \times (m_2 - 2)$ and $N \times (m_3 - u)$, matrices, respectively.

3.   Main Results

In the following Theorems we give only a sketch of the proofs and we leave all the technical details in the Appendix.

3.1. $4 \times 4 \times 4$ *saturated factorial design*

Theorem 3.1. *For the saturated $4 \times 4 \times 4$ factorial designs, with $N = 10$ runs, it holds that $|det(U_{d_{10}})| \leq 9$ and the design $d_{10}^* = \{001, 022, 030, 102, 113, 120, 203, 211, 310, 331\}$ is a D-optimal design.*

Proof. Consider the $4 \times 4 \times 4$ saturated factorial designs with $N = 10$ runs and corresponding matrix $U_{d_{10}}$ given by (2.1). If $n_i^s = 1$, $i = 1, 2, 3$ for some $s = 0, 1, 2, 3$, then, as noted at Remark 2.2, $|det(U_d)| = |det(U_{d'})|$, where d' is a saturated $3 \times 4 \times 4$ design with $|det(U_{d'})| \leq 7$, as proved by [4]. On the other hand, if $n_i^s \geq 2$, $i = 1, 2, 3$, $s = 0, 1, 2, 3$, then, according to Remark 2.7, we consider the cases $(n_i^0, n_i^1, n_i^2, n_i^3) = (4, 2, 2, 2)$ or $(n_i^0, n_i^1, n_i^2, n_i^3) = (3, 3, 2, 2)$. For the first case, according to Lemma 2.4, we have $|det(U_{d_{10}})| \leq 8$. Thus, we have to prove $|det(U_{d_{10}})| \leq 9$ for the case $(n_i^0, n_i^1, n_i^2, n_i^3) = (3, 3, 2, 2)$. For this case we consider the partition (2.3) of matrix $U_{d_{10}}$ with $u = 1$, because $n_1^0 = 3$ and $m_3 = 4$. The first column of matrices $Z_2^{(1)}$ and Z_3 have two or three $1's$.

- Assume that the first column of $Z_2^{(1)}$ and the first column of Z_3 contain both three $1's$. If the three $1's$ are all in block $Z_{2,12}$ or $Z_{3,12}$, respectively, then the first column of $Z_1^{(1)}$ and $Z_2^{(1)}$ or $Z_1^{(1)}$ and Z_3 are the same and, obviously $|det(U_{d_{10}})| = 0$. If these three $1's$ are two in block $Z_{2,1j}$ (or $Z_{3,1j}$) and one in block $Z_{2,1k}$ (or $Z_{3,1k}$), $j \neq k$, $j, k = 2, 3, 4$, then by subtracting this column from the $(j-1)$-th column of matrix $Z_1^{(1)}$ and expanding matrix $U_{d_{10}}$ along the first three columns we get $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 2| + |(-1) \cdot 1 \cdot 2| = 6$, for $j = 2$ and $k = 2$ or $k = 3$ or $|det(U_{d_{10}})| \leq |(-1) \cdot 2 \cdot 2| = 4$, for $j = 3$ or $j = 4$ and $k = 2$. Consequently, if one of the first columns of $Z_2^{(1)}$ or Z_3 contains three $1's$, then these $1's$ will be equally distributed at matrices $Z_{2,1j}$ or $Z_{3,1j}$, $j = 2, 3, 4$. In this case we have $|det(U_{d_{10}})| \leq 8$ (see Lemma A.2, available in Appendix).

- Assume that the first column of $Z_2^{(1)}$ (or Z_3) contains two $1's$. Then $|det(U_{d_{10}})| = 0$ if the two $1's$ of the first column of $Z_2^{(1)}$ (or Z_3) are all in $Z_{2,13}$ or $Z_{2,14}$ ($Z_{3,13}$ or $Z_{3,14}$), because this column of $Z_2^{(1)}$ (or Z_3) is identical with the second or the third column of $U_{d_{10}}$. If the two $1's$ are in block $Z_{2,12}$ (or $Z_{3,12}$) then by subtracting this column from the first column of matrix $Z_1^{(1)}$ and expanding matrix $U_{d_{10}}$ along the first three columns we get $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 2| = 4$.

- If the first column of $Z_2^{(1)}$ (or Z_3) contains two $1's$, one at matrix $Z_{i,13}$ and one 1 at matrix $Z_{i,14}$, $i = 2, 3$, then $|det(U_{d_{10}})| \leq 9$, according to Lemma A.3 (available in Appendix). Consequently, if at least one of the first columns of $Z_2^{(1)}$ (or Z_3) contains two $1's$ then one must be at $Z_{2,12}$ (or $Z_{3,12}$).

- Assume that the first column of $Z_2^{(1)}$ and the first column of Z_3 contain two $1's$ such that one is at matrix $Z_{i,12}$, $i = 2, 3$. Then, according to Lemma A.4 (available in Appendix), we have $|det(U_{d_{10}})| \leq 9$.

- If the first column of matrix Z_3 (or $Z_2^{(1)}$) contains two $1's$, one at matrix $Z_{3,12}$ (or $Z_{2,12}$) and the first column of matrix $Z_2^{(1)}$ (or Z_3) contains three $1's$, distributed

equally at matrices $Z_{2,1j}$ (or $Z_{3,1j}$), then, by Lemma A.5 (available in Appendix), we get $|\det(U_{d_{10}})| \leq 9$.

Consequently, a design is D-optimal in the class D_10 of all saturated $4 \times 4 \times 4$ designs, if $|\det(U_{d_{10}})| = 9$. One can easily verify that $|\det(U_{d_{10}^*})| = 9$, where $U_{d_{10}^*}$ is the matrix which corresponds to $d_{10}^* = \{001, 022, 030, 102, 113, 120, 203, 211, 310, 331\}$. So, d_{10}^* is a D-optimal saturated $4 \times 4 \times 4$ design.   ∎

3.2. $4 \times 4 \times 5$ *saturated factorial design*

Theorem 3.2. *For the saturated* $4 \times 4 \times 5$ *factorial designs, with* $N = 11$ *runs, it holds that* $|det(U_{d_{11}})| \leq 12$ *and the design* $d_{11}^* = \{002, 013, 024, 103,$ $111, 130, 214, 220, 231, 300, 322\}$ *is a D-optimal design.*

Proof. Consider the matrix $U_{d_{11}}$ given by (2.1), which corresponds to any saturated $4 \times 4 \times 5$ factorial design with $N = 11$ runs. If $n_3^s = 1$ for some $s = 0, 1, 2, 3, 4$, then, as noted at Remark 2.2, $|\det(U_{d_{11}})| = |\det(U_{d'})|$, where d' is a saturated $4 \times 4 \times 4$ design with $|\det(U_{d'})| \leq 9$, as proved in Theorem 3.1. On the other hand, if $n_i^s = 1$ for some $i = 1, 2$, $s = 0, 1, 2, 3$, then by Remark 2.2, we have $|\det(U_{d_{11}})| = |\det(U_{d'})|$, where d' is a saturated $3 \times 4 \times 5$ design with $|\det(U_{d'})| \leq 8$, as proved by [4]. If $n_3^s \geq 2$, then from (2.2) we get that $(n_3^0, n_3^1, n_3^2, n_3^3, n_3^4) = (3, 2, 2, 2, 2)$ and for $i = 1, 2$, $s = 0, 1, 2, 3$ that $(n_i^0, n_i^1, n_i^2, n_i^3) = (5, 2, 2, 2)$ or $(4, 3, 2, 2)$ or $(3, 3, 3, 2)$. Applying Lemma 2.4 for $i = 1$ or $i = 2$ we get $|\det(U_{d_{11}})| \leq 8$ for the first case and $|\det(U_{d_{11}})| \leq 12$ for the second case. So, in order to prove the theorem we deal only with the case $(n_i^0, n_i^1, n_i^2, n_i^3) = (3, 3, 3, 2)$ for $i = 1, 2$ and $(n_3^0, n_3^1, n_3^2, n_3^3, n_3^4) = (3, 2, 2, 2, 2)$. Matrix $U_{d_{11}}$ can be partitioned as in (2.3) with $u = 2$, because $n_1^0 = 3$ and $m_3 = 5$. The first column of matrix $Z_2^{(1)}$ contains two or three 1's, because $(n_2^0, n_2^1, n_2^2, n_2^3) = (3, 3, 3, 2)$, while the first two columns of matrix Z_3 contain either both two 1's, or one column (w.l.g. let the first) contains three 1's and the other column (let the second) contains two 1's, because $n_3^0 = 3$ and $n_3^s = 2$, $s = 1, 2, 3, 4$. Then considering all the cases, see Lemmas A.6-A.9, where the 1's are distributed in the first column of $Z_2^{(1)}$ and in the first two columns of Z_3, we prove that $|\det(U_{d_{11}})| \leq 12$. Therefore, a design is D-optimal in the class of the saturated $4 \times 4 \times 5$ designs if $|\det(U_{d_{11}})| \leq 12$. This bound is attained by $d_{11}^* = \{002, 013, 024, 103, 111, 130, 214, 220, 231, 300, 322\}$, because the corresponding matrix gives $|\det(U_{d_{11}^*})| = 12$.   ∎

3.3. $4 \times 4 \times m_3$, $m_3 \geq 6$ *saturated factorial design*

Theorem 3.3. *For the saturated* $4 \times 4 \times m_3$, $m_3 \geq 6$ *factorial designs, with* $N = m_3 + 6$ *runs, it holds that* $|det(U_{d_{m_3+6}})| \leq 17$.

Proof. Let $U_{d_{m_3+6}}$ given by (2.1), be the matrix which corresponds to any saturated $4 \times 4 \times m_3$, $m_3 \geq 6$ design with $N = m_3 + 6$ runs. Recalling Remark 2.2 we have $|\det(U_{d_{m_3+6}})| = |\det(U_{d_{12}})|$, where matrix $U_{d_{12}}$ corresponds to $4 \times 4 \times 6$ saturated design with $N = 12$ runs. If $n_3^s = 1$ for some $s = 0, 1, 2, 3, 4, 5$ then as noted at Remark 2.2 we have $|\det(U_{d_{12}})| = |\det(U_{d'})|$, where d' is a saturated $4 \times 4 \times 5$ design with $|\det(U_{d'})| \leq 12$, as proved in Theorem 3.2. If $n_i^s = 1$ for some $i = 1, 2$ and $s = 0, 1, 2, 3$ then by Remark 2.2 we have $|\det(U_{d_{12}})| = |\det(U_{d'})|$, where d' is a saturated $3 \times 4 \times 6$ design with $|\det(U_{d'})| \leq 8$, as proved by [4]. Obviously for the third factor we have $n_3^s = 2$,

$s = 0, 1, 2, 3, 4, 5$ and for $i = 1, 2$ the following cases arise: $(n_i^0, n_i^1, n_i^2, n_i^3) = (6, 2, 2, 2)$ or $(5, 3, 2, 2)$ or $(4, 4, 2, 2)$ or $(4, 3, 3, 2)$ or $(3, 3, 3, 3)$. Applying Lemma 2.4 for the first three cases we get: $|\det(U_{d_{12}})| \leq 8$ or $|\det(U_{d_{12}})| \leq 12$ or $|\det(U_{d_{12}})| \leq 16$, respectively. So we have to prove that $|\det(U_{d_{12}})| \leq 17$ for the other two cases.

- Case 1. Assume that $(n_i^0, n_i^1, n_i^2, n_i^3) = (4, 3, 3, 2)$ for $i = 1, 2$ and $n_3^s = 2$ for $s = 0, 1, 2, 3, 4, 5$. It holds that $|\det(U_{d_{12}})| \leq 16$ (see Lemma A.10, available in Appendix).
- Case 2. Assume that $(n_i^0, n_i^1, n_i^2, n_i^3) = (3, 3, 3, 3)$ for $i = 1, 2$ and $n_3^s = 2$ for $s = 0, 1, 2, 3, 4, 5$. It holds that $|\det(U_{d_{12}})| \leq 17$ (see Lemma A.11, available in Appendix).

Consequently, a saturated $4 \times 4 \times 6$ and hence $4 \times 4 \times m_3$, $m_3 \geq 6$ design is D-optimal if $|\det(U_{d_{m_3+6}})| = |\det(U_{d_{12}})| \leq 17$. Let $d_{m_3+6}^* = \{003, 024, 035, 036, \ldots, 03(m_3 - 1), 104, 110, 121, 211, 225, 232, 300, 312, 333\}$ be a saturated $4 \times 4 \times m_3$ design with $m_3 \geq 6$. One can easily verify that $|\det(U_{d_{m_3+6}^*})| = 16$. Although $|\det(U_{d_{m_3+6}^*})|$ does not attain the upper bound mentioned in Theorem 3.3, we conjecture that this is a D-optimal design. The D-efficiency of this design is $(16/17)^{1/m_3+6}$, that is $99, 5\%$ for $m_3 = 6$, because there are $m_3 + 6$ independent parameters in the model. ■

4.　Conclusions

In this article we dealt with D-optimal $4 \times 4 \times m_3$, $m_3 \geq 4$ saturated designs. For the general case of $4 \times m_2 \times m_3$, $m_2 \geq m_3 \geq 4$ saturated designs, useful Lemmas, Remarks and Notations on D-optimal designs were mentioned. The proof of the Theorems was based on the partitioning that the matrix corresponding to the Optimal design must have. In addition, the upper bounds of the determinant of the D-optimal saturated designs and the designs that achieve this bound were found for the cases of $4 \times 4 \times 4$ and $4 \times 4 \times 5$, while for the case $4 \times 4 \times m_3$, $m_3 \geq 6$ we presented a saturated design that we conjecture is D-optimal.

Appendix A. Appendix

The following notations and abbreviations are used in Appendix:
- w.l.g.: Without loss of generality.
- (A, j): j-th column of matrix A.
- $r(A, j)$: j-th column of matrix A which arises by adding or subtracting other columns of matrix A to matrix (A, j)
- rN: the resulting N-th column of a matrix which arises by expanding this matrix along some column.
- e.a.f.N.c.: expanding (or expand) a matrix along the first N columns.
- e.a.N.c.: expanding (or expand) a matrix along the N-th column.

Remark A.1. Consider equation (2.1). There is at most one row of U_d where the first column of $Z_2^{(1)}$ and some of the u first columns of Z_3 contain both one 1, otherwise by interchanging the 2nd and the 1st factor some of the cases described below arise.

The following four lemmas are used to prove Theorem 4, which concerns $4 \times 4 \times 4$ saturated factorial designs.

Lemma A.2. *Assume that the first column of $Z_2^{(1)}$ and Z_3 contains both three $1's$ distributed equally at $Z_{i,1j}$, $i = 2, 3$, $j = 2, 3, 4$ of (2.3). Then $|det(U_{d_{10}})| \leq 8$.*

Proof. We consider the following three cases.

- Case 1. The $1's$ of $Z_{2,1j}$ and $Z_{3,1j}$, $j = 2, 3, 4$ are all in different rows. Then $r(U_{d_{10}}, 1) = (U_{d_{10}}, 1) + (U_{d_{10}}, 2) + (U_{d_{10}}), 3) - (Z_2^{(1)}, 1) - (Z_3, 1)$. E.a.f.3.c. we get $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 2| = 4$.
- Case 2. The $1's$ of $Z_{2,12}$ and $Z_{3,12}$ are at the same row and the other two $1's$ of $Z_2^{(1)}$ and Z_3 are at different rows of $U_{d_{10}}$. Then $r(U_{d_{10}}, 3) = (U_{d_{10}}, 2) + (U_{d_{10}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1)$. E.a.f.3.c. we get $|det(U_{d_{10}})| \leq |(-2) \cdot 2 \cdot 2| = 8$.
- Case 3. The $1's$ of $Z_{2,13}$ (or $Z_{2,14}$) and $Z_{3,13}$ (or $Z_{3,14}$) are at the same row and the other two $1's$ of $Z_2^{(1)}$ and Z_3 are at different rows of $U_{d_{10}}$. Then $r(U_{d_{10}}, 1) = (U_{d_{10}}, 1) + (U_{d_{10}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1)$ (or $r(U_{d_{10}}, 1) = (U_{d_{10}}, 1) + (U_{d_{10}}, 2) - (Z_2^{(1)}, 1) - (Z_3, 1)$). E.a.f.3.c. we get $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 2| + |(-2) \cdot 1 \cdot 2| = 8$.

Consequently: $|det(U_{d_{10}})| \leq 8$. ∎

Lemma A.3. *If the first column of $Z_2^{(1)}$ (or Z_3) contains two $1's$, such that one at $Z_{2,13}$ (or $Z_{3,13}$) and one 1 at $Z_{2,14}$ (or $Z_{3,14}$) then $|det(U_{d_{10}})| \leq 9$.*

Proof. Assume w.l.g. that the first column of Z_3 contains one 1 at $Z_{3,13}$ and one 1 at $Z_{3,14}$. Then, by subtracting this column from the third column of $U_{d_{10}}$ and e.a.f.3.c. we have $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 3| + |(-1) \cdot 1 \cdot 3| = 9$. ∎

Lemma A.4. *Let the first column of $Z_2^{(1)}$ and Z_3 contain two $1's$, such that one is at $Z_{i,12}$, $i = 2, 3$. It holds that $|det(U_{d_{10}})| \leq 9$.*

Proof. We consider the following three cases.

- Case1. The $1's$ of the first column of $Z_2^{(1)}$ and Z_3 are all in different rows.
- Case1a. W.l.g. let the first column of $Z_2^{(1)}$ and Z_3 contain the second 1 at $Z_{2,13}$ and $Z_{3,13}$, respectively. Then $r(U_{d_{10}}, 1) = (U_{d_{10}}, 1) + (U_{d_{10}}, 2) - (Z_2^{(1)}, 1) - (Z_3, 1)$. E.a.f.3.c. we get $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 2| = 4$.
- Case 1b. W.l.g. let the first column of $Z_2^{(1)}$ and Z_3 contain the second 1 at $Z_{2,13}$ and $Z_{3,14}$, respectively. Then $r(U_{d_{10}}, 1) = (U_{d_{10}}, 1) - (Z_2^{(1)}, 1) - (Z_3, 1)$. E.a.f.3.c. we have $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 2| + |(-1) \cdot 1 \cdot 2| + |(-1) \cdot 1 \cdot 2| = 8$.
- Case 2. The $1's$ of the first column of $Z_{2,12}$ and $Z_{3,12}$ are at the same row.
- Case 2a. W.l.g. let the first column of $Z_2^{(1)}$ and Z_3 contain the other 1 at $Z_{2,13}$ and $Z_{3,13}$, respectively. Then $r(U_{d_{10}}, 2) = (U_{d_{10}}, 2) - (Z_2^{(1)}, 1) - (Z_3, 1)$. E.a.f.3.c. we get $|det(U_{d_{10}})| \leq |(-2) \cdot 2 \cdot 2| = 8$.
- Case 2b. W.l.g. let the first column of $Z_2^{(1)}$ and Z_3 contain the second 1 at $Z_{2,13}$ and $Z_{3,14}$, respectively. Then $r(U_{d_{10}}, 3) = (U_{d_{10}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1)$. E.a.f.3.c. we get $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 3| + |1 \cdot 1 \cdot 3| = 9$.
- Case 3. The $1's$ of the first column of $Z_{2,12}$ and $Z_{3,12}$ are at different rows and the $1's$ of $Z_{2,13}$ (or $Z_{2,14}$) and $Z_{3,13}$ (or $Z_{3,14}$) are at the same row. Then $r(U_{d_{10}}, 1) = (U_{d_{10}}, 1) - (Z_2^{(1)}, 1) - (Z_3, 1)$. E.a.f.3.c. we get $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 2| + |(-2) \cdot 1 \cdot 2| = 8$. ∎

Therefore: $|det(U_{d_{10}})| \leq 9$.

Lemma A.5. *Let the first column of Z_3 contains two $1's$, one at $Z_{3,12}$ and the*

first column of $Z_2^{(1)}$ contain three $1's$ distributed equally at $Z_{2,1j}$, $j = 2,3,4$. Then $|det(U_{d_{10}})| \leq 9$.

Proof. We consider the following three cases.

- Case 1. The $1's$ of the first column of $Z_2^{(1)}$ and Z_3 are all in different rows. Then $r(U_{d_{10}}, 1) = (U_{d_{10}}, 1) + (U_{d_{10}}, 2) + (U_{d_{10}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1)$ E.a.f.3.c. we get $|det(U_{d_{10}})| \leq |1 \cdot 2 \cdot 2| + |1 \cdot 1 \cdot 2| = 6$.

- Case 2. The $1's$ of the first column of $Z_{2,12}$ and $Z_{3,12}$ are at the same row. W.l.g.:

$$U_d = \left(\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & \\
\hline
1 & 0 & 0 & 1 & & 1 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & \\
\hline
0 & 1 & 0 & 1 & & 0 & \\
 & & & & Z_{2,23} & & Z_{3,23} \\
0 & 1 & 0 & 0 & & 1 & \\
\hline
0 & 0 & 1 & 1 & & 0 & \\
 & & & & Z_{2,24} & & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & \\
\end{array} \right).$$

E.a.2.c. we have $|det(U_{d_{10}})| \leq |det(U_7)| + |det(U_8)|$, where $|det(U_{d_i})|$, $i = 7,8$ is the minor of the element $(i, 2)$. For the quantity $|det(U_7)|$ we make the following: $r(U_7, 1) = (U_7, 1) - (U_7, 3)$. E.a.f.2.c. we get $|det(U_7)| \leq |2 \cdot 2| + |(-1) \cdot 1| = 5$ For the quantity $|det(U_8)|$ we make the following: $r(U_8, 1) = (U_8, 1) - (U_8, 6)$. E.a.f.2.c. we get $|det(U_8)| \leq |2 \cdot 2| = 4$, So, for case 2 we have $|det(U_{d_{10}})| \leq 9$.

- Case 3. The $1's$ of the first column of $Z_{2,13}$ (or $Z_{2,14}$) and $Z_{3,13}$ (or $Z_{3,14}$) are at the same row. W.l.g.:

$$U_{d_{10}} = \left(\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & \\
\hline
1 & 0 & 0 & 1 & & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & \\
\hline
0 & 1 & 0 & 1 & & 1 & \\
 & & & & Z_{2,23} & & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & \\
\hline
0 & 0 & 1 & 1 & & 0 & \\
 & & & & Z_{2,24} & & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & \\
\end{array} \right).$$

Following the procedure described in Case 2, we have $|det(U_{d_{10}})| \leq |det(U_7)| + |det(U_8)|$. For the quantity $|det(U_7)|$: $r(U_7, 1) = (U_7, 1) - (U_7, 3) - (U_7, 6)$. E.a.f.2.c.

we get $|\det(U_7)| \leq |1 \cdot 2| + |(-1) \cdot 1| = 3$. For the quantity $|\det(U_8)|$ e.a.f.2.c. we get $|\det(U_8)| \leq 6$. So, for case 3 we have $|\det(U_{d_{10}})| \leq 9$.

From cases 1, 2 and 3 we conclude that $|\det(U_{d_{10}})| \leq 9$.   ∎

For the proof of Theorem 5 the following four lemmas are necessary. Also, for their proof, the partitioning of matrix $U_{d_{11}}$ of $4 \times 4 \times 5$ saturated designs, as stated in (2.1) and (2.3), must be taken into account.

Lemma A.6. *If at least one column of $U_{d_{11}}$ exists, the first of $Z_2^{(1)}$ (or one of the first two columns of Z_3), which contains two or three $1's$ all at $Z_{2,1j}$ (or $Z_{3,1j}$), $j = 2, 3, 4$ then $|det(U_{d_{11}})| \leq 6$ or ($|det(U_{d_{11}})| = 0$).*

Proof. If the first column of $Z_2^{(1)}$ (or one of the first two columns of Z_3) contains two $1's$, all in $Z_{2,14}$ (or $Z_{3,14}$), then this column is identical with the third column of $U_{d_{11}}$ and $|\det(U_{d_{11}})| = 0$. Similarly, if the first column of $Z_2^{(1)}$ (or one of the first two columns of Z_3) contains three $1's$, all in $Z_{2,1j}$ (or $Z_{3,1j}$), then this can only happen for $j = 2, 3$. In this case, this column is identical with the $(j-1)$-th column of $U_{d_{11}}$ and $|\det(U_{d_{11}})| = 0$. If the first column of $Z_2^{(1)}$ (or one of the first two columns of Z_3) contains two $1's$, all in $Z_{2,1j}$ (or $Z_{3,1j}$), $j = 2, 3$, then, by subtracting this column from the $(j - 1)$-th, $j = 2, 3$ column of matrix $U_{d_{11}}$ and expanding $U_{d_{11}}$ along its first three columns we get $|\det(U_{d_{11}})| \leq |1 \cdot 3 \cdot 2| = 6$.   ∎

Lemma A.7. *Assume that the first of $Z_2^{(1)}$ (or one of the first two columns of Z_3) contains three $1's$, one at $Z_{2,1j}$ (or $Z_{3,1j}$), and two at $Z_{2,1k}$ (or $Z_{3,1k}$) $j, k = 2, 3, 4$, $j \neq k$, then $|det(U_{d_{11}})| \leq 10$.*

Proof. Subtracting this column from the $(j-1)$-th column of matrix $U_{d_{11}}$ and expanding $U_{d_{11}}$ along its first three columns we get $|\det(U_{d_{11}})| \leq |(-1) \cdot 3 \cdot 2| = 6$ if $j = 4$ or $|\det(U_{d_{11}})| \leq |1 \cdot 3 \cdot 2| + |(-1) \cdot 2 \cdot 2| = 10$, if $j = 2, 3$.   ∎

Lemma A.8. *Assume that the first two columns of Z_3 contain two $1's$, one at $Z_{3,1j}$ and one 1 at $Z_{3,1k}$, $j, k = 2, 3, 4$, $j \neq k$,. Moreover, if the first column of matrix $Z_2^{(1)}$ contains*
- *two $1's$, one at matrix $Z_{2,1j}$ and one 1 at matrix $Z_{2,1k}$, $j, k = 2, 3, 4$, $j \neq k$, or*
- *three $1's$, where each of the matrices $Z_{2,1j}$, $j = 2, 3, 4$ contains one 1,*

then $|det(U_{d_{11}})| \leq 12$.

Proof. We consider the following cases:
- Case 1. Each of the first two columns of Z_3 contains one 1 at $Z_{3,12}$ and one 1 at $Z_{3,13}$. Then $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |1 \cdot 3 \cdot 2| + |1 \cdot 2 \cdot 2| = 10$.
- Case 2. Each of the first two columns of Z_3 contains one 1 at $Z_{3,12}$ (or $Z_{3,13}$) and one 1 at $Z_{3,14}$. Then $r(U_{d_{11}}, 3) = (U_{d_{11}}, 1) + (U_{d_{11}}, 3) - (Z_3, 1) - (Z_3, 2)$ (or $r(U_{d_{11}}, 3) = (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_3, 1) - (Z_3, 2)$). E.a.f.3.c. we have $|\det(U_{d_{11}})| \leq |1 \cdot 2 \cdot 3| = 6$.
- Case 3. The first column of Z_3 contains one 1 at $Z_{3,12}$ and one 1 at $Z_{3,14}$ and the second column of Z_3 contains one 1 at $Z_{3,13}$ and one 1 at $Z_{3,14}$. Then $r(U_{d_{11}}, 3) = (U_{d_{11}}, 3) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we have $|\det(U_{d_{11}})| \leq |(-1) \cdot 2 \cdot 3| + |(-1) \cdot 2 \cdot 3| =$

12.

- **Case 4.** The first column of Z_3 contains one 1 at $Z_{3,12}$ and one 1 at $Z_{3,13}$ and the second column of Z_3 contains one 1 at $Z_{3,12}$ ($Z_{3,13}$) and one 1 at $Z_{3,14}$ or the contrary. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(\begin{array}{ccc|cc|ccc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & & & 1 & 0 & \\
1 & 0 & 0 & Z_{2,12} & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & & & 0 & 0 & \\
\hline
0 & 1 & 0 & & & 1 & 0 & \\
0 & 1 & 0 & Z_{2,13} & Z_{2,23} & 0 & 0 & Z_{3,23} \\
0 & 1 & 0 & & & 0 & 0 & \\
\hline
0 & 0 & 1 & Z_{2,14} & Z_{2,24} & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & & & 0 & 0 & \\
\end{array}\right).
$$

Consider now the first column of $Z_2^{(1)}$ which contains two 1's as described in the first bullet of Lemma A.8. The following cases arise:

- **Case 4Aa.** The first column of $Z_2^{(1)}$ contains one 1 at $Z_{2,12}$ and one 1 at $Z_{2,13}$. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(\begin{array}{ccc|cc|ccc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & & & 1 & 0 & \\
1 & 0 & 0 & \text{one ace at } Z_{2,12} & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & & & 0 & 0 & \\
\hline
0 & 1 & 0 & & & 1 & 0 & \\
0 & 1 & 0 & \text{one ace at } Z_{2,13} & Z_{2,23} & 0 & 0 & Z_{3,23} \\
0 & 1 & 0 & & & 0 & 0 & \\
\hline
0 & 0 & 1 & 0 & Z_{2,24} & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & 0 & \\
\end{array}\right).
$$

- **Case 4Aai.** There is no row in matrix $U_{d_{11}}$ in which the first column of $Z_2^{(1)}$ and the first two columns of Z_3 simultaneously contain an 1. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |1 \cdot 2 \cdot 2| + |1 \cdot 1 \cdot 3| = 7$.

- **Case 4Aaii.** There is one row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and only one of the two first two columns of Z_3 (let the first) contain both one 1.

Assume that this row is one of the n_1^1 rows. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \begin{pmatrix} 0 & 0 & 0 & 0 & & 0 & 0 & \\ 0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\ 0 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 1 & 0 & 0 & 1 & & 1 & 0 & \\ 1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\ 1 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 1 & 0 & 0 & & 1 & 0 & \\ 0 & 1 & 0 & 1 & Z_{2,23} & 0 & 0 & Z_{3,23} \\ 0 & 1 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 0 & 1 & 0 & & 0 & 1 & \\ 0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 & Z_{3,24} \end{pmatrix}.$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$: $r(U_{10}, 1) = (U_{10}, 1) - (U_{10}, 6) - (U_{10}, 7)$. E.a.f.2.c. we get $|\det(U_{10})| \leq |1 \cdot 3| + |(-1) \cdot 2| = 5$. For the quantity $|\det(U_{11})|$: $r(U_{11}, 2) = (U_{11}, 2) - (U_{11}, 3) - (U_{11}, 6)$. E.a.f.2.c. we get $|\det(U_{11})| \leq |1 \cdot 3| + |(-2) \cdot 2| = 7$. Consequently, for this case we have $|\det(U_{d_{11}})| \leq 12$.

Assume that this row is one of the n_1^2 rows. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we have $|\det(U_{d_{11}})| \leq |(-2) \cdot 2 \cdot 3| + |(-1) \cdot 1 \cdot 3| = 11$.

- Case 4Aaiii. There is one row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the first column of Z_3 contain both one 1 and another row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the second column of Z_3 contain both one 1. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \begin{pmatrix} 0 & 0 & 0 & 0 & & 0 & 0 & \\ 0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\ 0 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 1 & 0 & 0 & 0 & & 1 & 0 & \\ 1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 & Z_{3,22} \\ 1 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 1 & 0 & 1 & & 1 & 0 & \\ 0 & 1 & 0 & 0 & Z_{2,23} & 0 & 0 & Z_{3,23} \\ 0 & 1 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 0 & 1 & 0 & & 0 & 1 & \\ 0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 & Z_{3,24} \end{pmatrix}.$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$ by

expanding successively along the columns 7, $r3$ and $r5$ we get we get $|\det(U_{10})| \leq |1 \cdot 2| = 2$. For the quantity $|\det(U_{11})|$ by e.a.f.2.c. we get $|\det(U_{11})| \leq |3 \cdot 3| = 9$. Hence, $|\det(U_{d_{11}})| \leq 11$.

- Case 4Ab. The first column of $Z_2^{(1)}$ contains one 1 at $Z_{2,12}$ and one 1 at $Z_{2,14}$. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \begin{pmatrix} 0 & 0 & 0 & & 0 & & & 0 & 0 & \\ 0 & 0 & 0 & & 0 & & Z_{2,21} & 0 & 0 & Z_{3,21} \\ 0 & 0 & 0 & & 0 & & & 0 & 0 & \\ \hline 1 & 0 & 0 & & & & & 1 & 0 & \\ 1 & 0 & 0 & \text{one ace at } Z_{2,12} & & Z_{2,22} & 0 & 1 & Z_{3,22} \\ 1 & 0 & 0 & & & & & 0 & 0 & \\ \hline 0 & 1 & 0 & & 0 & & & 1 & 0 & Z_{3,23} \\ 0 & 1 & 0 & & 0 & & Z_{2,23} & 0 & 0 & \\ 0 & 1 & 0 & & 0 & & & 0 & 0 & \\ \hline 0 & 0 & 1 & & & & & 0 & 1 & Z_{3,24} \\ 0 & 0 & 1 & \text{one ace at } Z_{2,14} & & Z_{2,24} & 0 & 0 & \end{pmatrix}.$$

- Case 4Abi. There is no row in matrix $U_{d_{11}}$ in which the first column of $Z_2^{(1)}$ and the first two columns of Z_3 simultaneously contain an 1. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |1 \cdot 2 \cdot 2| + |1 \cdot 2 \cdot 2| = 8$.

- Case 4Abii. There is one row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the second column of Z_3 contain both one 1.

  Assume that this row is one of the n_1^1 rows. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \begin{pmatrix} 0 & 0 & 0 & 0 & & 0 & 0 & \\ 0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\ 0 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 1 & 0 & 0 & 0 & & 1 & 0 & \\ 1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 & Z_{3,22} \\ 1 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 1 & 0 & 0 & & 1 & 0 & Z_{3,23} \\ 0 & 1 & 0 & 0 & Z_{2,23} & 0 & 0 & \\ 0 & 1 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 0 & 1 & 0 & Z_{2,24} & 0 & 1 & Z_{3,24} \\ 0 & 0 & 1 & 1 & & 0 & 0 & \end{pmatrix}.$$

Following the procedure described in case 4Aaii we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| +$

$|\det(U_{11})|$. One can easily verify that $|\det(U_i)| \leq |2 \cdot 3| = 6$, $i = 10, 11$, by subtracting the seventh column of U_{10} from the first and the third column of U_{11} from the first and e.a.f.2.c. So, $|\det(U_{d_{11}})| \leq 12$.

Assume that this row is one of the n_1^3 rows. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we have $|\det(U_{d_{11}})| \leq |(-1) \cdot 2 \cdot 2| + |(-2) \cdot 1 \cdot 3| = 10$.

- Case 4Abiii. The statement is identical with that of case 4Aaiii. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \left(\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 1 & & 1 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 0 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 1 & & 0 & 1 & \\
0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 & Z_{3,24}
\end{array} \right).$$

Evaluating $|\det(U_{d_{11}})|$ in a similar way to that mentioned in the case 4Aaiii we get, $|\det(U_{d_{11}})| \leq 11$.

- Case 4Ac. The first column of $Z_2^{(1)}$ contains one 1 at $Z_{2,13}$ and one 1 at $Z_{2,14}$. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \left(\begin{array}{ccc|c|cc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \quad Z_{2,21} & 0 & 0 \quad Z_{3,21} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \quad Z_{2,22} & 0 & 1 \quad Z_{3,22} \\
1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & \text{one ace at } Z_{2,13} & 1 & 0 \\
0 & 1 & 0 & \qquad\qquad Z_{2,23} & 0 & 0 \quad Z_{3,23} \\
0 & 1 & 0 & & 0 & 0 \\
\hline
0 & 0 & 1 & \text{one ace at } Z_{2,14} \quad Z_{2,24} & 0 & 1 \\
0 & 0 & 1 & & 0 & 0 \quad Z_{3,24}
\end{array} \right).$$

- Case 4Aci. There is no row in matrix $U_{d_{11}}$ in which the first column of $Z_2^{(1)}$ and the first two columns of Z_3 simultaneously contain an 1. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) +$

$(U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |1 \cdot 3 \cdot 2| + |1 \cdot 2 \cdot 2| = 10$.

- Case 4Acii. There is one row in matrix $U_{d_{11}}$ in which the first column of $Z_2^{(1)}$ and the second column of Z_3 contain both one 1. Necessarily this row is one of the n_1^3 rows of $U_{d_{11}}$. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \left(\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & Z_{3,23} \\
0 & 1 & 0 & 1 & Z_{2,23} & 0 & 0 & \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 1 & Z_{2,24} & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & 0 & \\
\end{array} \right).$$

Following the procedure described in case 4Aaii we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$ expand along the seventh column and then along the $r3$ column. Subtracting $r5$ from $r1$ and e.a.f.2.c. we get $|\det(U_{10})| \leq |1 \cdot 2| + |(-1) \cdot 1| = 3$ For the quantity $|\det(U_{11})|$, e.a.f.2.c. we get $|\det(U_{11})| \leq |3 \cdot 3| = 9$. Hence, $|\det(U_{d_{11}})| \leq 12$.

- Case 4Aciii. The statement is identical with that of case 4Aaii. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \left(\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 1 & & 1 & 0 & Z_{3,23} \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 0 & \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 1 & Z_{2,24} & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & 0 & \\
\end{array} \right).$$

Following the procedure described in case 4Aaii we have $|\det(U_{d_{11}})| \leq 11$.

Consider now the first column of $Z_2^{(1)}$ which contains three 1's as described in the second bullet of Lemma 10. The following cases arise:

- Case 4Bai. The 1's of first column of $Z_2^{(1)}$ and the second column of Z_3 are in different rows. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |1 \cdot 3 \cdot 2| + |(-1) \cdot 2 \cdot 2| = 10$.

- Case 4Baii.There is one row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the second column of Z_3 contain both one 1.

    Assume that this row is one of the n_1^1 rows. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(
\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & \\
0 & 1 & 0 & 1 & Z_{2,23} & 0 & 0 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 0 & Z_{2,24} & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & 1 & & 0 & 0 & \\
\end{array}
\right).
$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$: $r(U_{10}, 1) = (U_{10}, 1) - (U_{10}, 7)$. E.a.f.2.c. we get $|\det(U_{10})| \leq |2 \cdot 3| = 6$. For the quantity $|\det(U_{11})|$: $r(U_{11}, 1) = (U_{11}, 1) + (U_{11}, 2) - (U_{11}, 3) - (U_{11}, 6)$. E.a.f.2.c. we get $|\det(U_{11})| \leq |1 \cdot 3| + |1 \cdot 2| = 5$. Consequently, for this case we have $|\det(U_{d_{11}})| \leq 11$.

    Assume that this row is one of the n_1^3 rows. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |1 \cdot 2 \cdot 2| + |(-1) \cdot 1 \cdot 3| + |1 \cdot 1 \cdot 3| = 10$. Hence, for this case we have $|\det(U_{d_{11}})| \leq 11$.

- Case 4Baiii.There is one row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the first column of Z_3 contain both one 1 and another row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the second column of Z_3 contain both one 1.

    Assume that this row is one of the n_1^1 rows and the other is one of the n_1^3 rows.

W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \left(\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 1 & & 1 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & \\
0 & 1 & 0 & 1 & Z_{2,23} & 0 & 0 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 1 & Z_{2,24} & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & 0 &
\end{array} \right).$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$ expand along the seventh column, then subtract $r3$ and $r6$ from $r2$ and finally e.a.f.2.c. We get $|\det(U_{10})| \leq |1 \cdot 2| + |(-2) \cdot 1| = 4$. For the quantity $|\det(U_{11})|$: $r(U_{11}, 1) = (U_{11}, 1) - (U_{11}, 6)$. E.a.f.2.c. we get $|\det(U_{11})| \leq |2 \cdot 3| + |(-1) \cdot 2| = 8$. Consequently, for this case we have $|\det(U_{d_{11}})| \leq 12$.

Assume that this row is one of the n_1^2 rows and the other is one of the n_1^1 rows. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \left(\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 1 & & 1 & 0 & \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 0 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 0 & Z_{2,24} & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & 1 & & 0 & 0 &
\end{array} \right).$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$ expand along the seventh column, subtract $r6$ from $r1$ and finally e.a.f.2.c. We get $|\det(U_{10})| \leq |1 \cdot 3| + |(-1) \cdot 2| = 5$. For the quantity $|\det(U_{11})|$: $r(U_{11}, 1) = (U_{11}, 1) - (U_{11}, 3) - (U_{11}, 6)$. E.a.f.2.c. we get $|\det(U_{11})| \leq |1 \cdot 3| + |(-2) \cdot 2| = 7$. Consequently, for this case we have $|\det(U_{d_{11}})| \leq 12$.

Assume that this row is one of the n_1^2 rows and the other is one of the n_1^3 rows.

W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(
\begin{array}{ccc|cc|cc c}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 1 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 1 & & 1 & 0 & \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 0 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 1 & & 0 & 1 & \\
0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 & Z_{3,24} \\
\end{array}
\right).
$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$ by expanding along the seventh column, subtracting $r3$ and $r6$ from $r1$ and finally e.a.f.2.c. we get $|\det(U_{10})| \leq |(-2) \cdot 2| = 4$. For the quantity $|\det(U_{11})|$: $r(U_{11}, 1) = (U_{11}, 1) - (U_{11}, 6)$. E.a.f.2.c. we get $|\det(U_{11})| \leq |2 \cdot 3| + |(-1) \cdot 2| = 8$. Consequently, for this case we have $|\det(U_{d_{11}})| \leq 12$.

From all the above cases we finally get that $|\det(U_{d_{11}})| \leq 12$. ■

Lemma A.9. *Let the second column of matrix Z_3 contain two $1's$, one at $Z_{3,1j}$ and one 1 at $Z_{3,1k}$, $j, k = 2, 3, 4$, $j \neq k$ and the first column of matrix Z_3 contain three $1's$, where each of the matrices $Z_{3,1j}$, $j = 2, 3, 4$ contains one 1. Moreover, if the first column of matrix $Z_2^{(1)}$ contains*

- *two $1's$, one at matrix $Z_{2,1j}$ and one 1 at matrix $Z_{2,1k}$, $j, k = 2, 3, 4$, $j \neq k$, or*
- *three $1's$, where each of the matrices $Z_{2,1j}$, $j = 2, 3, 4$ contains one 1,*

then $|det(U_{d_{11}})| \leq 12$.

Proof. We consider the following cases:

- Case 1. The second column of matrix Z_3 contains one 1 at $Z_{3,12}$ (or $Z_{3,13}$) and one 1 at $Z_{3,14}$. Then $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 3) - (Z_3, 1) - (Z_3, 2)$ (or $r(U_{d_{11}}, 2) = (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_3, 1) - (Z_3, 2)$). E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |1 \cdot 3 \cdot 2| + |(-1) \cdot 2 \cdot 2| = 10$.
- Case 2. The second column of matrix Z_3 contains one 1 at matrix $Z_{3,12}$ and one

1 at $Z_{3,13}$. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & & 0 & & 0 & 0 & \\
0 & 0 & 0 & & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & & & & 1 & 0 & \\
1 & 0 & 0 & Z_{2,12} & Z_{2,22} & & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & & & & 0 & 0 & \\
\hline
0 & 1 & 0 & & & & 1 & 0 & \\
0 & 1 & 0 & Z_{2,13} & Z_{2,23} & & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & & & & 0 & 0 & \\
\hline
0 & 0 & 1 & & & & 1 & 0 & \\
0 & 0 & 1 & Z_{2,14} & Z_{2,24} & & 0 & 0 & Z_{3,24}
\end{array}\right).
$$

Consider now the first column of $Z_2^{(1)}$ which contains two 1's as described in the first bullet of Lemma A.9. The following cases arise:

- Case 2Aa. The first column of $Z_2^{(1)}$ contains one 1 at $Z_{2,12}$ and one 1 at $Z_{2,13}$. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(\begin{array}{ccc|ccc|cc}
0 & 0 & 0 & & 0 & & 0 & 0 \\
0 & 0 & 0 & & 0 & Z_{2,21} & 0 & 0 \; Z_{3,21} \\
0 & 0 & 0 & & 0 & & 0 & 0 \\
\hline
1 & 0 & 0 & & & & 1 & 0 \\
1 & 0 & 0 & \text{one ace at } Z_{2,12} & Z_{2,22} & & 0 & 1 \; Z_{3,22} \\
1 & 0 & 0 & & & & 0 & 0 \\
\hline
0 & 1 & 0 & & & & 1 & 0 \\
0 & 1 & 0 & \text{one ace at } Z_{2,13} & Z_{2,23} & & 0 & 1 \; Z_{3,23} \\
0 & 1 & 0 & & & & 0 & 0 \\
\hline
0 & 0 & 1 & & 0 & & 1 & 0 \\
0 & 0 & 1 & & 0 & Z_{2,24} & 0 & 0 \; Z_{3,24}
\end{array}\right).
$$

- Case 2Aai. There is no row in matrix $U_{d_{11}}$ in which the first column of $Z_2^{(1)}$ and the first two columns of Z_3 simultaneously contain an 1. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3)) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |(-1) \cdot 1 \cdot 3| = 3$.

- Case 2Aaii. There is one row of $U_{d_{11}}$ where the first column of matrix $Z_2^{(1)}$ and the second column of Z_3 contain both one 1. Assume, w.l.g. that this row is one of the n_1^1 rows of $U_{d_{11}}$. Matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \begin{pmatrix} 0 & 0 & 0 & 0 & & 0 & 0 & \\ 0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\ 0 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 1 & 0 & 0 & 0 & & 1 & 0 & \\ 1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 & Z_{3,22} \\ 1 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 1 & 0 & 0 & & 1 & 0 & \\ 0 & 1 & 0 & 0 & Z_{2,23} & 0 & 1 & Z_{3,23} \\ 0 & 1 & 0 & 1 & & 0 & 0 & \\ \hline 0 & 0 & 1 & 0 & & 1 & 0 & \\ 0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 & Z_{3,24} \end{pmatrix}.$$

Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3)) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \le |(-1) \cdot 2 \cdot 2| + |1 \cdot 2 \cdot 2| + |1 \cdot 1 \cdot 3| = 11$.

- Case 2Aaiii. There is one row of $U_{d_{11}}$ where the first column of matrix $Z_2^{(1)}$ and the second column of Z_3 contain both one 1 and another row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the second column of Z_3 contain both one 1. Wl.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \begin{pmatrix} 0 & 0 & 0 & 0 & & 0 & 0 & \\ 0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\ 0 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 1 & 0 & 0 & 1 & & 1 & 0 & \\ 1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\ 1 & 0 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 1 & 0 & 0 & & 1 & 0 & \\ 0 & 1 & 0 & 1 & Z_{2,23} & 0 & 1 & Z_{3,23} \\ 0 & 1 & 0 & 0 & & 0 & 0 & \\ \hline 0 & 0 & 1 & 0 & & 1 & 0 & \\ 0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 & Z_{3,24} \end{pmatrix}.$$

E.a.1.c. we have $|\det(U_{d_{11}})| \le |\det(U_4)| + |\det(U_5)| + |\det(U_6)|$. For the quantity $|\det(U_4)|$ by expanding along the third column, subtracting $r5$ from $r1$ and e.a.f.2.c. we get $|\det(U_4)| \le |1 \cdot 2| + |(-1) \cdot 1| = 3$. For the quantity $|\det(U_5)|$ by expanding along the seventh column, then along $r3$, subtracting $r5$ from $r1$ and e.a.f.2.c. we have $|\det(U_5)| \le |1 \cdot 2| + |(-1) \cdot 1| = 3$. For the quantity $|\det(U_6)|$ e.a.f.2.c. we get $|\det(U_6)| \le |3 \cdot 2| = 6$. Hence, $|\det(U_{d_{11}})| \le 12$.

- Case 2Ab. The first column of matrix $Z_2^{(1)}$ acontains one 1 at $Z_{2,12}$ (or $Z_{2,13}$) and one 1 at $Z_{2,14}$.Wl.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \begin{pmatrix}
0 & 0 & 0 & & 0 & & & 0 & 0 & \\
0 & 0 & 0 & & 0 & & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & & 0 & & & 0 & 0 & \\
\hline
1 & 0 & 0 & & & & & 1 & 0 & \\
1 & 0 & 0 & \text{one ace at } Z_{2,12} & & Z_{2,22} & & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & & & & & 0 & 0 & \\
\hline
0 & 1 & 0 & & 0 & & & 1 & 0 & \\
0 & 1 & 0 & & 0 & & Z_{2,23} & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & & 0 & & & 0 & 0 & \\
\hline
0 & 0 & 1 & & & & & 1 & 0 & \\
0 & 0 & 1 & \text{one ace at } Z_{2,14} & & Z_{2,24} & & 0 & 0 & Z_{3,24}
\end{pmatrix}.
$$

- Case 2Abi. There is no row in matrix $U_{d_{11}}$ in which the first column of $Z_2^{(1)}$ and the first two columns of Z_3 simultaneously contain an 1. Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \le |1 \cdot 2 \cdot 2| = 4$.
- Case 2Abii. There is one row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the first column of Z_3 contain both one 1. This is Case 4Baii from Lemma A.8, for which $|\det(U_{d_{11}})| \le 11$.
- Case 2Abiii. The statement is the same with the statement of Case 4Aaiii, that is there is one row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the first column of Z_3 contain both one 1 and another row of $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the second column of Z_3 contain both one 1. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \begin{pmatrix}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 1 & Z_{2,24} & 1 & 0 & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & 0 &
\end{pmatrix}.
$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$ by expanding along the third column, then along $r6$, subtracting $r5$ from $r1$ and e.a.f.2.c. we get $|\det(U_{10})| \leq |1 \cdot 2| + |(-1) \cdot 1| = 3$. For the quantity $|\det(U_{11})|$ e.a.f.2.c. we get $|\det(U_{11})| \leq |3 \cdot 3| = 9$. Hence $|\det(U_{d_{11}})| \leq 12$.

Consider now the first column of $Z_2^{(1)}$ which contains three $1's$ as described in the second bullet of Lemma A.9. The following cases arise:

- Case 2Bai. There is no row of matrix $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the first column of Z_3 contain both one 1. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & 1 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 0 & & 1 & 0 & \\
0 & 0 & 1 & 1 & Z_{2,24} & 0 & 0 & Z_{3,24} \\
\end{array}\right).
$$

Then, $r(U_{d_{11}}, 3) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1)$. E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |1 \cdot 2 \cdot 3| + |1 \cdot 2 \cdot 3| = 12$.

- Case 2Baii. There is one row of matrix $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the first column of Z_3 contain both one 1.

  Assume that this row is one of the n_1^1 (or n_1^2) rows. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 1 & & 1 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & \\
0 & 1 & 0 & 1 & Z_{2,23} & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 0 & & 1 & 0 & \\
0 & 0 & 1 & 1 & Z_{2,24} & 0 & 0 & Z_{3,24} \\
\end{array}\right).
$$

Then, $r(U_{d_{11}}, 3) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$.
E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |(-1) \cdot 2 \cdot 3| + |1 \cdot 2 \cdot 3| = 12$.

Assume that this row is one of the n_1^3 rows. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \left(\begin{array}{ccc|cc|cc c}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 1 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & 1 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 1 & & 1 & 0 & \\
0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 & Z_{3,24} \\
\end{array} \right).$$

Then, $r(U_{d_{11}}, 1) = (U_{d_{11}}, 1) + (U_{d_{11}}, 2) + (U_{d_{11}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2)$.
E.a.f.3.c. we get $|\det(U_{d_{11}})| \leq |(-1) \cdot 3 \cdot 1| + |1 \cdot 3 \cdot 1| = 6$. Consequently, for this case we have $|\det(U_{d_{11}})| \leq 12$.

- Case 2Baiii. There is one row of matrix $U_{d_{11}}$ where the first column of $Z_2^{(1)}$ and the first column of Z_3 contain both one 1 and another row of $U_{d_{11}}$ (let w.l.g. a row of the n_1^1 rows) where the first column of $Z_2^{(1)}$ and the second column of Z_3 contain both one 1.

  Assume that the first of the above rows is one of n_1^2 rows. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$U_{d_{11}} = \left(\begin{array}{ccc|cc|cc c}
0 & 0 & 0 & 0 & & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & \\
1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 1 & 0 & 1 & & 1 & 0 & \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & \\
\hline
0 & 0 & 1 & 0 & & 1 & 0 & \\
0 & 0 & 1 & 1 & Z_{2,24} & 0 & 0 & Z_{3,24} \\
\end{array} \right).$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$:
$r(U_{10}, 1) = r(U_{10}, 1) + r(U_{10}, 2) - r(U_{10}, 6) - r(U_{10}, 7)$. E.a.f.2.c. we get $|\det(U_{10})| \leq$

$|1 \cdot 3| + |1 \cdot 2| = 5$. For the quantity $|\det(U_{11})|$: $r(U_{11}, 2) = r(U_{11}, 2) - r(U_{11}, 3) - r(U_{11}, 7)$. E.a.f.2.c. we get $|\det(U_{11})| \leq |(-2) \cdot 2| + |1 \cdot 3| = 7$. Hence $|\det(U_{d_{11}})| \leq 12$.

Assume that the first of the above rows is one of n_1^3 rows. W.l.g. matrix $U_{d_{11}}$ can be written as:

$$
U_{d_{11}} = \left(
\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & & 0 & 0 \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 \quad Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 \\
1 & 0 & 0 & 1 & Z_{2,22} & 0 & 1 \quad Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 1 \quad Z_{3,23} \\
0 & 1 & 0 & 1 & & 0 & 0 \\
\hline
0 & 0 & 1 & 1 & & 1 & 0 \\
0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 \quad Z_{3,24}
\end{array}
\right).
$$

E.a.3.c. we have $|\det(U_{d_{11}})| \leq |\det(U_{10})| + |\det(U_{11})|$. For the quantity $|\det(U_{10})|$: $r(U_{10}, 1) = r(U_{10}, 1) + r(U_{10}, 2) - r(U_{10}, 3) - r(U_{10}, 6) - r(U_{10}, 7)$. E.a.f.2.c. we get $|\det(U_{10})| \leq |(-1) \cdot 2| + |1 \cdot 2| = 4$. For the quantity $|\det(U_{11})|$: $r(U_{11}, 1) = r(U_{11}, 1) - r(U_{11}, 7)$. E.a.f.2.c. we get $|\det(U_{11})| \leq |2 \cdot 3| + |(-1) \cdot 2| = 8$. Hence $|\det(U_{d_{11}})| \leq 12$.

From all the above cases we conclude that $|\det(U_{d_{11}})| \leq 12$. ∎

In order to prove Theorem 6, which concerns $4 \times 4 \times 6$ saturated factorial designs, we need the following two lemmas.

Lemma A.10. *Consider the saturated* $4 \times 4 \times 6$ *design* d_{12} *and assume moreover that* $(n_i^0, n_i^1, n_i^2, n_i^3) = (4, 3, 3, 2)$ *for* $i = 1, 2$ *and* $n_3^s = 2$ *for* $s = 0, 1, 2, 3, 4, 5$. *Then* $|det(U_{d_{12}})| \leq 16$.

Proof. Matrix $U_{d_{12}}$, given by (2.1) can be partitioned as in (2.3), with $u = 2$, because $n_1^0 = 4$ and $m_3 = 6$. The first two columns of Z_3 contain both two 1's, because $n_3^s = 2$ for $s = 0, 1, 2, 3, 4, 5$. We consider the following three cases.

- Case 1. At least one of the two first columns of matrix Z_3 contains the two 1's all at matrix $Z_{3,1j}$, $j = 2, 3, 4$. By subtracting this column from the $(j - 1)$-th column of matrix $U_{d_{12}}$ and e.a.f.3.c.we get $|\det(U_{d_{12}})| \leq |1 \cdot 3 \cdot 2| = 6$ for $j = 2, 3$ or $|\det(U_{d_{12}})| = 0$ for $j = 4$, becasue this column is identical with the third column of $U_{d_{12}}$.

- Case 2. At least one of the two first columns of Z_3 contains one 1 at $Z_{3,12}$ and one 1 at $Z_{3,13}$. By subtracting this column from the first column of $U_{d_{12}}$ and e.a.f.3.c we have $|\det(U_{d_{12}})| \leq |2 \cdot 3 \cdot 2| + |(-1) \cdot 2 \cdot 2| = 16$.

- Case 3. At least one of the two first columns of Z_3 contains one 1 at $Z_{3,12}$ ($Z_{3,13}$) and one 1 at $Z_{3,14}$. By subtracting this column from the third column of $U_{d_{12}}$ and e.a.f.3.c we have $|\det(U_{d_{12}})| \leq |1 \cdot 3 \cdot 3| + |(-1) \cdot 2 \cdot 3| = 15$.

Therefore: $|\det(U_{d_{12}})| \leq 16$. ∎

Lemma A.11. *Consider the saturated* $4 \times 4 \times 6$ *design* d_{12}. *Let* $(n_i^0, n_i^1, n_i^2, n_i^3) = (3,3,3,3)$ *for* $i = 1,2$ *and* $n_3^s = 2$ *for* $s = 0,1,2,3,4,5$. *It holds that* $|\det(U_{d_{12}})| \leq 17$.

Proof. Matrix $U_{d_{12}}$, given by (2.1) can be partitioned as in (2.3), with $u = 3$, because $n_1^0 = 4$ and $m_3 = 6$. The first two columns of $Z_2^{(1)}$ contain three 1's, because $n_2^s = 3$, while each of the first three columns of Z_3 contain two 1's, because $n_3^s = 2$ for $s = 0,1,2,3,4,5$. We consider the following three cases.

- Case 1. There is at least one column, w.l.g. the first of $Z_2^{(1)}$ (or one of the first three of Z_3), which contains three (or two) 1's all at $Z_{2,1j}$ ($Z_{3,1j}$), $j = 2,3,4$. Then $|\det(U_{d_{12}})| = 0$ if this column is one of the first two columns of $Z_2^{(1)}$. If this column is one of the first three columns of Z_3, then by subtracting this column from the $(j-1)$-th column of $U_{d_{12}}$ and expanding along its first three columns we get $|\det(U_{d_{12}})| \leq |1 \cdot 3 \cdot 3| = 9$.

- Case 2. Let one of the first two columns of $Z_2^{(1)}$ (w.l.g. the first) contain two 1's at $Z_{2,1j}$ and one 1 at $Z_{2,1k}$, where $j, k = 2,3,4$, $j \neq k$. Then by subtracting this column from the $(j-1)$-th column of $U_{d_{12}}$ and expanding along its first three columns we get $|\det(U_{d_{12}})| \leq |1 \cdot 3 \cdot 3| + |(-1) \cdot 2 \cdot 3| = 15$.

- Case 3. Now assume that the first three columns of Z_3, which each contains two 1's, such that, one is at $Z_{3,1j}$ and one is at $Z_{3,1k}$, $j, k = 2,3,4$, $j \neq k$ and one (w.l.g. let the first) column of $Z_2^{(1)}$ contains three 1's, distributed equally at $Z_{2,1j}$, $j = 2,3,4$. Then:

- Case 3a. Two of the three first columns of Z_3 contain one 1 at matrix $Z_{3,1i}$ and one 1 at matrix $Z_{3,1j}$, $i, j = 2,3,4$, $i \neq j$. Then add the $(j-1)$-th column of $U_{d_{12}}$ to the $(i-1)$-th column and subtract from the resulting column the first two columns of Z_3. Expanding $U_{d_{12}}$ along the first three columns we get $|\det(U_{d_{12}})| \leq |1 \cdot 3 \cdot 3| + |1 \cdot 2 \cdot 3| = 15$.

- Case 3b. From what was mentioned previously we conclude that none of the matrices $Z_{3,12}$, $Z_{3,13}$ or $Z_{3,14}$ can contain three 1's of the first three columns of Z_3. Hence, matrix $U_{d_{12}}$ can be written, w.l.g., as:

$$
U_{d_{11}} = \left(
\begin{array}{ccc|cc|ccc}
0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & 0 \quad Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & & & 1 & 0 & 0 \\
1 & 0 & 0 & Z_{2,12} & Z_{2,22} & 0 & 1 & 0 \quad Z_{3,22} \\
1 & 0 & 0 & & & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & & & 1 & 0 & 0 \\
0 & 1 & 0 & Z_{2,13} & Z_{2,23} & 0 & 0 & 1 \quad Z_{3,23} \\
0 & 1 & 0 & & & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & & Z_{2,24} & 0 & 1 & 0 \quad Z_{3,24} \\
0 & 0 & 1 & Z_{2,14} & & 0 & 0 & 1 \\
0 & 0 & 1 & & & 0 & 0 & 0 \\
\end{array}
\right).
$$

Let us now consider the first column of $Z_2^{(1)}$ which contains three 1's one in each

of the matrices $Z_{2,1j}$, $j = 2, 3, 4$. The following cases arise:

- Case 3bi. The 1's of matrices $Z_{2,1j}$ and $Z_{3,1j}$, $j = 2, 3, 4$ are all at different rows. The column, which arises by adding the first three columns of Z_3 to the first column of $Z_2^{(1)}$ is identical with the column which arises by adding the second and the third column of $U_{d_{12}}$ to the first column. So, $|\det(U_{d_{12}})| = 0$.

- Case 3bii. There is one row of $U_{d_{12}}$ where the first column of $Z_2^{(1)}$ and only one of the first three columns of Z_3 contain both one 1. W.l.g. let this row be one of the n_1^3 rows of $U_{d_{12}}$. Then $r(U_{d_{12}}, 1) = (U_{d_{12}}, 1) + (U_{d_{12}}, 2) + (U_{d_{12}}, 3) - (Z_2^{(1)}, 1) - (Z_3, 1) - (Z_3, 2) - (Z_3, 3)$. E.a.f.3.c. we get $|\det(U_{d_{12}})| \leq |(-1) \cdot 2 \cdot 3| + |1 \cdot 2 \cdot 3| = 12$.

- Case 3biii. There are two rows of $U_{d_{12}}$ where the first column of $Z_2^{(1)}$ and two of the first three columns of Z_3 contain both one 1. W.l.g. let these rows be one of the n_1^2 and one of the n_1^3 rows of $U_{d_{12}}$. W.l.g. matrix $U_{d_{12}}$ can be written as:

$$
U_{d_{11}} = \left(
\begin{array}{ccc|cc|ccc|c}
0 & 0 & 0 & 0 & & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & \\
\hline
1 & 0 & 0 & 0 & & 1 & 0 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & 0 & Z_{3,22} \\
1 & 0 & 0 & 1 & & 0 & 0 & 0 & \\
\hline
0 & 1 & 0 & 1 & & 1 & 0 & 0 & \\
0 & 1 & 0 & 0 & Z_{2,23} & 0 & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & 0 & \\
\hline
0 & 0 & 1 & 0 & & 0 & 1 & 0 & \\
0 & 0 & 1 & 1 & Z_{2,24} & 0 & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & 0 & 0 & \\
\end{array}
\right) .
$$

E.a.3.c. we have $|\det(U_{d_{12}})| \leq |\det(U_{10})| + |\det(U_{11})| + |\det(U_{12})|$. For the quantity $|\det(U_{10})|$ by expanding along the seventh column, subtracting $r6$ from $r1$ and finally e.a.f.2.c. we get $|\det(U_{10})| \leq |1 \cdot 3| + |(-1) \cdot 2| = 5$. For the quantity $|\det(U_{11})|$ by expanding along the eighth column, subtracting $r3$ and $r6$ from $r1$ and finally e.a.f.2.c. we get $|\det(U_{11})| \leq |1 \cdot 2| + |(-2) \cdot 1| = 4$. For the quantity $|\det(U_{12})|$ we make the following: $r(U_{12}, 1) = (U_{12}, 1) - (U_{12}, 6)$. E.a.f.2.c. we get $|\det(U_{12})| \leq |2 \cdot 3| + |(-1) \cdot 2| = 8$. So, for case 3biii we have $|\det(U_{d_{12}})| \leq 17$.

- Case 3biv. There are three rows of $U_{d_{12}}$ where the first column of $Z_2^{(1)}$ and the first three columns of Z_3 contain both one 1. W.l.g. matrix $U_{d_{12}}$ can be written

as:

$$
U_{d_{11}} = \left(\begin{array}{ccc|cc|ccc|c}
0 & 0 & 0 & 0 & & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & Z_{2,21} & 0 & 0 & 0 & Z_{3,21} \\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & \\
\hline
1 & 0 & 0 & 1 & & 1 & 0 & 0 & \\
1 & 0 & 0 & 0 & Z_{2,22} & 0 & 1 & 0 & Z_{3,22} \\
1 & 0 & 0 & 0 & & 0 & 0 & 0 & \\
\hline
0 & 1 & 0 & 0 & & 1 & 0 & 0 & \\
0 & 1 & 0 & 1 & Z_{2,23} & 0 & 0 & 1 & Z_{3,23} \\
0 & 1 & 0 & 0 & & 0 & 0 & 0 & \\
\hline
0 & 0 & 1 & 1 & & 0 & 1 & 0 & \\
0 & 0 & 1 & 0 & Z_{2,24} & 0 & 0 & 1 & Z_{3,24} \\
0 & 0 & 1 & 0 & & 0 & 0 & 0 &
\end{array}\right).
$$

E.a.1.c. we have $|\det(U_{d_{12}})| \le |\det(U_4)| + |\det(U_5)| + |\det(U_6)|$. For the quantity $|\det(U_4)|$ by expanding along the sixth column, subtracting $r3$ and $r7$ from $r2$ and finally e.a.f.2.c. we get $|\det(U_4)| \le |1 \cdot 2| + |(-2) \cdot 1| = 4$. For the quantity $|\det(U_5)|$ by expanding along the seventh column and then along $r2$ we have $|\det(U_5)| \le |\det(U_{5,9})| + |\det(U_{5,10})|$, where $U_{5,i}$, $i = 9,10$ is the minor of the element $(i,2)$ of matrix U_5. For the quantity $|\det(U_{5,9})|$ by expanding along the sixth column, then along $r2$ and finally along $r4$, that is, the columns of $U_{5,9}$, that each contain one 1, we have $|\det(U_{5,9})| \le 1$. For the quantity $|\det(U_{5,10})|$ e.a.1.c. we get $|\det(U_{5,10})| \le 3$. Consequently, $|\det(U_5)| \le 4$. For the quantity $|\det(U_6)|$: $r(U_6,1) = (U_6,1) - (U_6,8)$. E.a.f.2.c. we get $|\det(U_6)| \le |2 \cdot 3| + |(-1) \cdot 2| = 8$. So, for case 3biv we have $|\det(U_{d_{12}})| \le 16$ and for case 3b we get $|\det(U_{d_{12}})| \le 17$. From all the above cases we conclude that $|\det(U_{d_{12}})| \le 17$. ∎

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